Probability Theory 8/25/22

Events $\leftrightarrow$ Subsets

Ω sample space
set of all possible outcome

$\mathcal{F}$ space of events

$A \in \mathcal{F} \Rightarrow A \subset \Omega$

A B

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$ = outcome is odd

$B = \{1, 2, 3\}$ = outcome ≤ 3

A^c = all elements of Ω not in A

$A^c = \{2, 4, 6\}$ not

$A \cap B = \{1, 3\}$ and

$A \cup B = \{1, 2, 3, 5\}$ or

$A \setminus B = \{5\} = A \cap B^c$

$\{5\}$ event 5 not an event

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$A^c = \Omega \setminus A \qquad \neg A \qquad \overline{A}$$

$\mathcal{F}$: not empty

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$A_i \in \mathcal{F} \quad i = 1 \dots \infty \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

Properties:

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \Rightarrow A \cup A^c \in \mathcal{F} \Rightarrow$$

$$\Omega \in \mathcal{F}$$

$$A_i \in \mathcal{F} \qquad i = 1 \dots n$$

$$A_i = A_n \qquad i > n$$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$$

$$\Omega \in \mathcal{F} \Rightarrow \Omega^c \in \mathcal{F} \Rightarrow \emptyset \in \mathcal{F}$$

$A_i \in \mathcal{F}$ $\qquad \bigcap_i A_i = (\bigcup A_i^c)^c$

$A_i^c \in \mathcal{F} \Rightarrow \bigcup A_i^c \in \mathcal{F} \Rightarrow (\bigcup A_i^c)^c \in \mathcal{F}$

$\mathcal{F}$ contains ϕ and Ω

$\mathcal{F}$ is closed for the complent

$\mathcal{F}$ is closed for countable unions and intersections

I is called countable

There exists a bijective function from $\mathbb{N}$ to I.

Examples:

$\mathcal{F} = \{\Omega, \phi\}$

$\mathcal{F} = 2^{\Omega}$ power set

Ω is finite

$A \subset \Omega$ $\qquad$ $\mathcal{F} = \{\Omega, A, A^c, \phi\}$

$\mathcal{G} = \{(a, b) \in \mathbb{R}\}$

$\mathcal{F}$ be the smallest space of event that contains $\mathcal{G}$.

Ω is finite or countable

ω elements of Ω

$\{\omega\}$ atomic event

$$A = \bigcup_{\omega \in A} \{\omega\}$$

Probability

$\mathbb{P}$ is a function from $\mathcal{F} \to \mathbb{R}$

$\mathbb{P}(A) \geq 0$

not necessary

$\boxed{\mathbb{P}(\phi) = 0}$ and $\mathbb{P}(\Omega) = 1$

$A_i \in \mathcal{F} \quad i = 1 \dots \infty$

$A_i \cap A_j = \phi \quad i \neq j$ disjoint
mutually exclusive

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

$A_i \quad i = 1 \dots n$ are disjoint

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

$A \quad A^c \qquad A \cap A^c = \phi$

$A \cup A^c = \Omega$

$$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega)$$

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

$A = \Omega \implies \mathbb{P}(\phi) = 0$

Definition

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Ω = sample space

$\mathcal{F}$ = space of events

$\mathbb{P}$ = probability

Probability Space

Ω

$\mathcal{F} = \{\Omega, A, A^c, \phi\}$

$\mathbb{P}(A) = p \qquad \mathbb{P}(A^c) = 1 - p$

Bernoulli situation

Ω is finite

$\mathcal{F} = 2^{\Omega}$

$\mathbb{P}(\{\omega\}) = p(\omega)$

$$\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$$

Probability of atomic events defines $\mathbb{P}$.

$$p(\omega) \geq 0 \qquad \sum_{\omega \in \Omega} p(\omega) = 1$$

Same Think works for countable sets.

Ω is finite:

all ω are equiprobable

$$p(\omega) = p(\omega') \quad \forall \omega, \omega'$$

$$p(\omega) = \frac{1}{|\Omega|}$$

$$\Omega = \mathbb{N} = \{0, 1, 2 \ldots\}$$

$$p(\omega) = 0 \quad \text{if} \quad \omega > 10$$

$$p(n) = p^n (1-p)$$

coin That gives H prob p.

How long will I wait before the first Tail?

IT happen at The first flip : $(1-p)$

2nd flip : $p(1-p)$

$p(n) = p^n(1-p)$ Geometric dist.

$$\sum_{n=0}^{\infty} p^n (1-p) = 1$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x|<1}$$

$$\mathbb{P}(A) = \sum_{n \in A} p(n)$$

$$\mathbb{P}(\text{even}) = \sum_{n \text{ even}} p^n (1-p) = \quad n = 2m$$

$$\sum_m p^{2m}(1-p) = \frac{(1-p)}{(1-p^2)}$$

A is a subset of $\mathbb{N}$

$A \cap \{0, 1, \ldots n\} = A_n$

$\mathbb{P}_n(A) = |A_n| / n$

if $\lim_{n \to \infty} \mathbb{P}_n(A) = \mathbb{P}(A)$ exists

I can call $\mathbb{P}(A)$ The prob of A.

$\mathcal{F}$ all subset of $\mathbb{N}$ for which The limit exists, $\Rightarrow$ $\mathcal{F}$ space of events

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$[0, 1] = \Omega$

(a, b) $\qquad \mathbb{P}((a, b)) = b - a$

$A = \bigcup_n (a_n, b_n)$ $\qquad b_n \leq a_{n+1}$

$\mathbb{P}(A) = \sum_n |b_n - a_n|$

$(a, b)^c = [0, a] \cup [b, 1]$

$(a, b)^c \cap \left(\frac{a+b}{2}, 1\right] = [b, 1]$